Exercise 19

In Exercises 1–26, solve the following Volterra integral equations by using the *Adomian decomposition method*:

$$u(x) = x^2 + \int_0^x (x - t)u(t)\, dt$$

Solution

Assume that $u(x)$ can be decomposed into an infinite number of components.

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

Substitute this series into the integral equation.

$$\sum_{n=0}^{\infty} u_n(x) = x^2 + \int_0^x (x - t) \sum_{n=0}^{\infty} u_n(t)\, dt$$

$$u_0(x) + u_1(x) + u_2(x) + \cdots = x^2 + \int_0^x (x - t)[u_0(t) + u_1(t) + \cdots]\, dt$$

$$u_0(x) + u_1(x) + u_2(x) + \cdots = \underbrace{x^2}_{u_0(x)} + \underbrace{\int_0^x (x - t)u_0(t)\, dt}_{u_1(x)} + \underbrace{\int_0^x (x - t)u_1(t)\, dt}_{u_2(x)} + \cdots$$

If we set $u_0(x)$ equal to the function outside the integral, then the rest of the components can be deduced in a recursive manner. After enough terms are written, a pattern can be noticed, allowing us to write a general formula for $u_n(x)$. Note that the $(x - t)$ in the integrand essentially means that we integrate the function next to it twice.

$$u_0(x) = x^2$$

$$u_1(x) = \int_0^x (x - t)u_0(t)\, dt = \int_0^x (x - t)(t^2)\, dt = \frac{x^4}{4 \cdot 3}$$

$$u_2(x) = \int_0^x (x - t)u_1(t)\, dt = \int_0^x (x - t)\left(\frac{t^4}{4 \cdot 3}\right) dt = \frac{x^6}{6 \cdot 5 \cdot 4 \cdot 3}$$

$$u_3(x) = \int_0^x (x - t)u_2(t)\, dt = \int_0^x (x - t)\left(\frac{t^6}{6 \cdot 5 \cdot 4 \cdot 3}\right) dt = \frac{x^8}{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}$$

$$\vdots$$

$$u_n(x) = \int_0^x (x - t)u_{n-1}(t)\, dt = \frac{2x^{2n+2}}{(2n + 2)!}$$

Therefore,

$$u(x) = \sum_{n=0}^{\infty} \frac{2x^{2n+2}}{(2n + 2)!} = 2\sum_{n=0}^{\infty} \frac{x^{2(n+1)}}{[2(n + 1)]!} = 2\sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!} = 2(\cosh x - 1).$$